Indian Statistical Institute

Midterm Examination 2017-2018

M.Math First Year   Functional Analysis   March 02, 2018   Instructor : Jaydeb Sarkar

Time : 3 Hours   Maximum Marks : 100

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(1) (10 marks) Let X and Y be normed linear spaces, and let $T : X \to Y$ be a linear operator. Prove that $T \in \mathcal{B}(X, Y)$ if and only if

$$\sup\{\|Tx\| : x \in X, \|x\| \leq 1\} < \infty.$$

(2) (10 marks) Prove that the closed unit ball of c_0 does not have any extreme points.

(3) (15 marks) True or false (with proper justification):
(i) l^∞ has a Schauder basis.
(ii) $C[0, 1]$ is separable.
(iii) $l^3 \setminus l^2 \neq \emptyset$.

(4) (15 marks) Let $\{a_n\} \in l^\infty$. Consider the linear map $T : l^\infty \to l^\infty$ defined by

$$T(\{x_n\}) = \{a_1 x_1, a_2 x_2, \ldots\} \qquad (\{x_n\} \in l^\infty).$$

Prove that T is bounded. Also compute the norm of T.

(5) (15 marks) Compute the dual of l^2.

(6) (15 marks) Let X and Y be Banach spaces, and let $\mathcal{D}$ be a dense subset of X. Let $\{T_n\} \subseteq \mathcal{B}(X, Y)$ be a uniformly bounded sequence (that is, $\sup_n \|T_n\| < \infty$). If $\{T_n x\} \subseteq Y$ is a Cauchy sequence for all $x \in \mathcal{D}$, then prove that there exists $T \in \mathcal{B}(X, Y)$ such that $T_n x \to Tx$, as $n \to \infty$, for all $x \in X$.

(7) (15 marks) Find a sequence of functions that converge in $L^1[0, 1]$ but not in $L^\infty[0, 1]$.

(8) (15 marks) Let $\{a_n\} \subseteq \mathbb{C}$ be a sequence. Suppose that the series $\sum a_n x_n$ is convergent for all $\{x_n\} \in c_0$. Prove that $\{a_n\} \in l^1$.